The zero set of the independence polynomial of a graph

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Complexity of numerical computation

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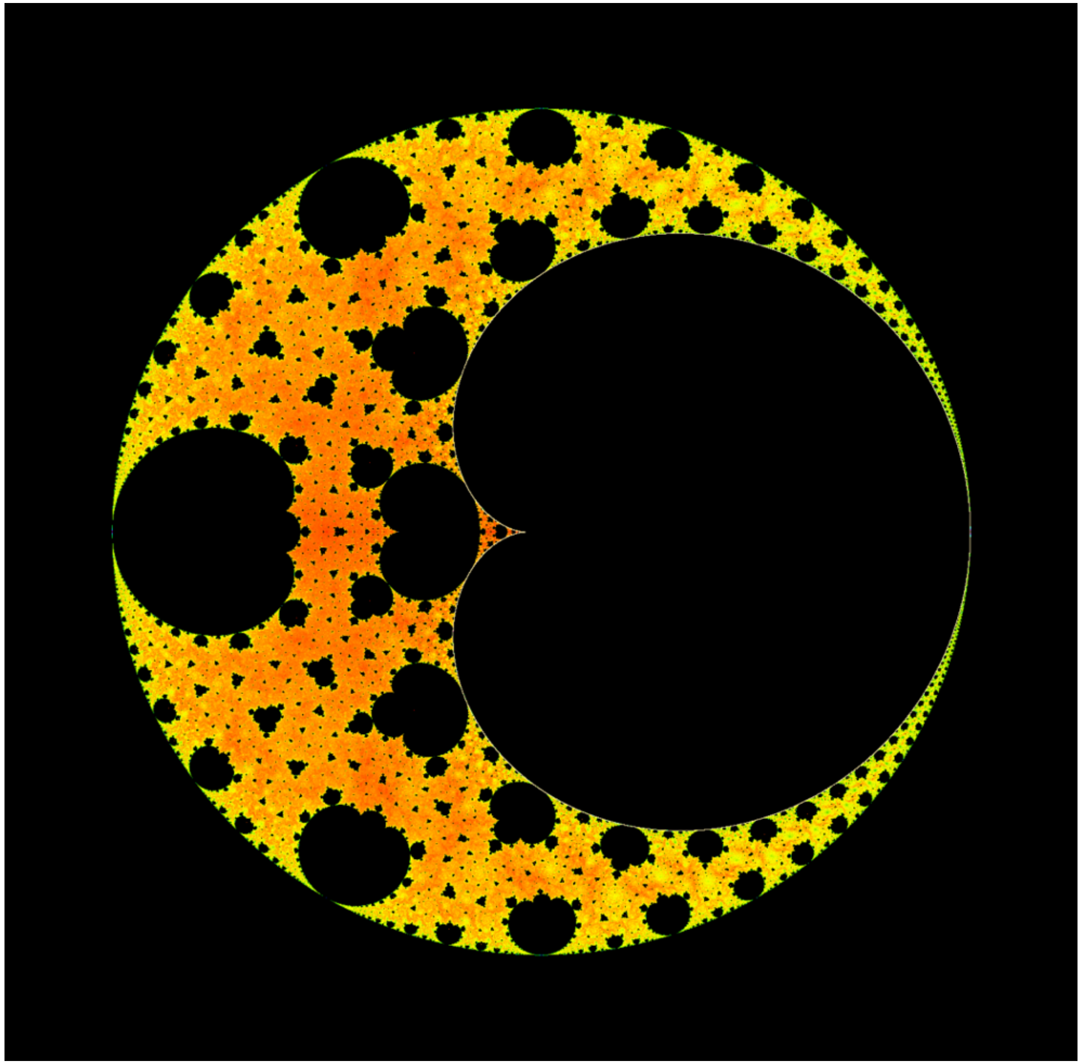


Figure by N. Fagella

INDEPENDENCE POLYNOMIALS OF GRAPHS

G finite graph

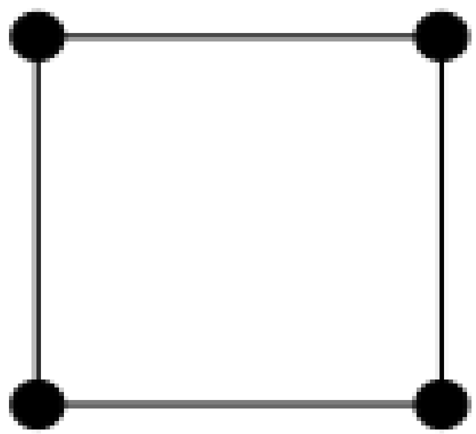
$I \subset V_G$ independent set if $v \not\sim v'$ $\forall v, v' \in I$

vertices of G

The independence polynomial of G is

$$Z_G = \sum_{\substack{I \subset V_G \\ \text{indep}}} x^{\# I} \in \mathbb{Z}_{\geq 0}[x]$$

Ex



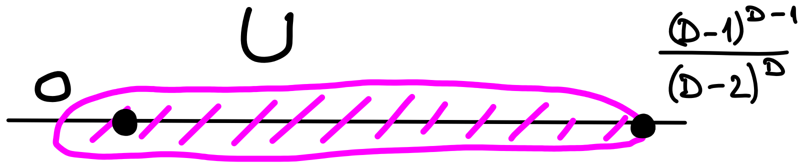
$$Z_G = 1 + 4x + 2x^2$$

$\mathcal{G}$ <u>countably infinite graph</u> of degree $D \in [3, +\infty)$

↑ mxl valence of nodes

THM (Peters & Regts 2019 after conjecture by Sokal 2001)

$\exists\, U \supset \left[0, \frac{(D-1)^{D-1}}{(D-2)^D}\right)$ open s.t. $Z_G(\lambda) \neq 0 \quad \forall\, G \subset \mathcal{G}$ finite



Q: Is there a limit distribution for the zeros of Z_G?

More precisely, for $F \in \mathbb{C}[x] \setminus \mathbb{C}$ set

$$\delta_F = \frac{1}{\deg(F)} \sum_{\lambda} \mathrm{mult}_\lambda(F) \cdot \delta_\lambda \qquad \text{discrete proba measure} / \mathbb{C}$$

$\exists\, \mu_{\mathcal{G}}$ proba measure$/\mathbb{C}$ s.t. $\lim_{G \to \mathcal{G}} \delta_{Z_G} = \mu_{\mathcal{G}}$ weak-$*$?

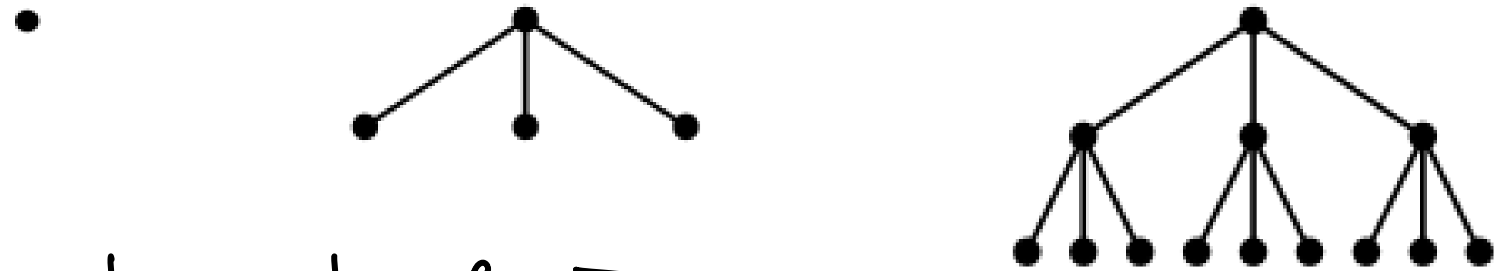
Regular rooted trees

For $d \geq 2$: $\mathcal{Z}$ regular rooted tree with branching d

For $k \geq -1$: $\mathcal{Z}_k$ subtree at depth $\leq k$

Ex: $d=3$

$k=0$ $\qquad$ $k=1$ $\qquad$ $k=2$



$T_k := z_{\mathcal{Z}_k}$ indep poly of $\mathcal{Z}_k$

$$R_k := \frac{T_k}{T_{k-1}^d} \in \mathbb{Q}(x)$$

then $T_{-1} = 1$, $T_0 = 1 + x$ & $T_k = T_{k-1}^d + x \cdot T_{k-2}^d$ for $k \geq 1$

and $R_0 = 1 + x$ & $R_k = 1 + \dfrac{x}{R_{k-1}^d}$ for $k \geq 1$

Complex dynamics

For $\lambda \in \mathbb{C}^\times$ $\quad f_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \quad z \mapsto 1 + \frac{\lambda}{z^d}$

Riemann sphere

- Critical points: $\{0, \infty\}$
- $0 \longrightarrow \infty \longrightarrow 1 \longrightarrow 1+\lambda$
- $f_\lambda^{-1}(\infty) = \{0\}$

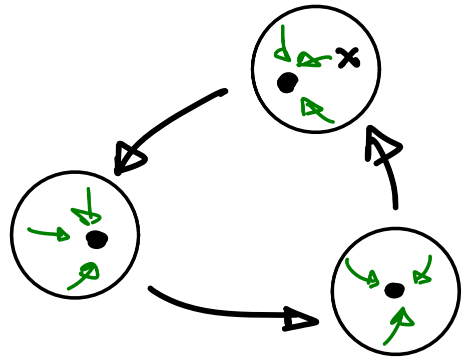
Hence $R_k(\lambda) = f_\lambda^k(1+\lambda) = f_\lambda^{k+3}(0)$, where $f^k = \underbrace{f \circ \cdots \circ f}_{k}$

$$\Omega_k := (T_k = 0) = \{\lambda \in \mathbb{C}^\times \mid f_\lambda^{k+3}(0) = 0\}$$

$$\Omega := \bigcup_{k \geq 0} \Omega_k = \{\lambda \in \mathbb{C}^\times \mid 0 \in \mathrm{Per}(f_\lambda)\}$$

HYPERBOLIC COMPONENTS AND ZERO FREE REGIONS

$\lambda \in \mathbb{C}^\times$ hyperbolic if the critical points of f_λ converge to attracting cycles



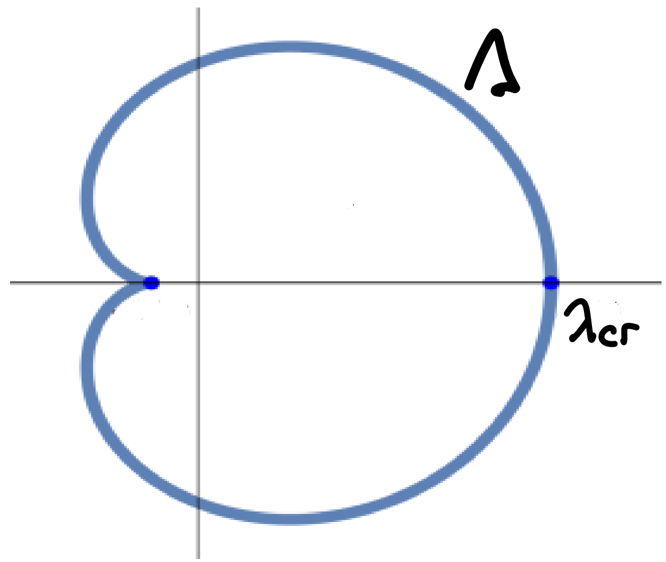
In our case:

λ hyperbolic iff f_λ has an attracting cycle (Fatou's theorem)

$\Rightarrow$ if λ is attracting but not superattracting then $\lambda \notin \Omega$

$\Lambda := \{\lambda \mid f_\lambda \text{ has } \underline{\text{attracting fixed point}}\}$

$$= \left\{ \frac{-d^d \alpha}{(d+\alpha)^{d+1}} \;\middle|\; \alpha \in \mathbb{D} \right\}$$



$\Longrightarrow \Lambda \cap \Omega = \emptyset$ (Peters & Regts 2019)

A "BIFURCATION" HEIGHT

For $\lambda \in \overline{\mathbb{Q}}^{\times}$ set

$$h(\lambda) := \lim_{k\to\infty} \frac{h_{\mathrm{Weil}}(f_\lambda^k(0))}{\deg(f_\lambda^k)} \in \mathbb{R}_{\geq 0} \qquad (\text{if well-defn})$$

THM (RL-S)

This limit defines a function $h : \overline{\mathbb{Q}}^{\times} \longrightarrow \mathbb{R}_{\geq 0}$ with

$$h(\lambda) = 0 \quad \text{iff} \quad 0 \in \mathrm{PrePer}(f_\lambda)$$

Moreover $h(\lambda) = \rho(\lambda) + \sum_{p \text{ prime}} \log^+|\lambda|_p$ with $\rho : \mathbb{C}^{\times} \longrightarrow \mathbb{R}$

potential of μ_{bif}, the normalized bifurcation measure

of $(f_\lambda)_{\lambda\in\mathbb{C}^{\times}}$, and $\rho - \log^+$ is continuous on $\mathbb{C}$

QUANTITATIVE EQUIDISTRIBUTION

h is a (quasicanonical) height function (Arakelov geometry)

+ arithmetic equidistribution thm (Favre & RL 2006)

Wasserstein distance

<u>COR</u> $\exists\, c>0$ st $W(\delta_{T_k}, \mu_{bif}) \leq c\left(\frac{\log k}{d^k}\right)^{1/2}$

I.e. $\forall f : \mathbb{C} \to \mathbb{R}$ Lipschitz

$$\left| \frac{1}{\# \Omega_k} \sum_{\lambda \in \Omega_k} f(\lambda) - \int f \, d\mu_{bif} \right| \leq c \left(\frac{\log k}{d^k}\right)^{1/2} \mathrm{Lip}(f)$$

In particular $\lim_{k \to \infty} \delta_{T_k} = \mu_{bif}$

$d = 2$

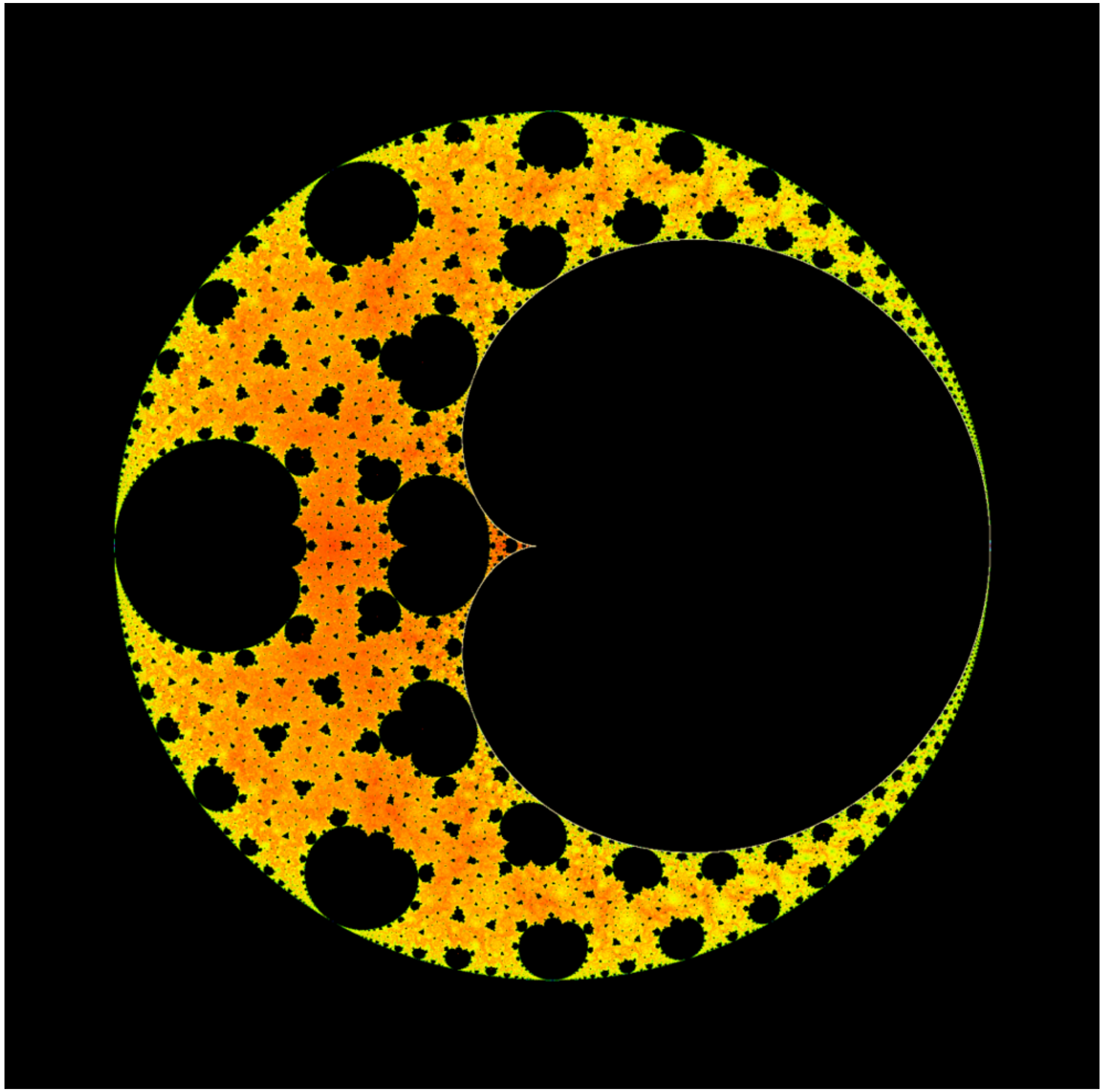


FIGURE BY N. FAGELLA

$d = 3$

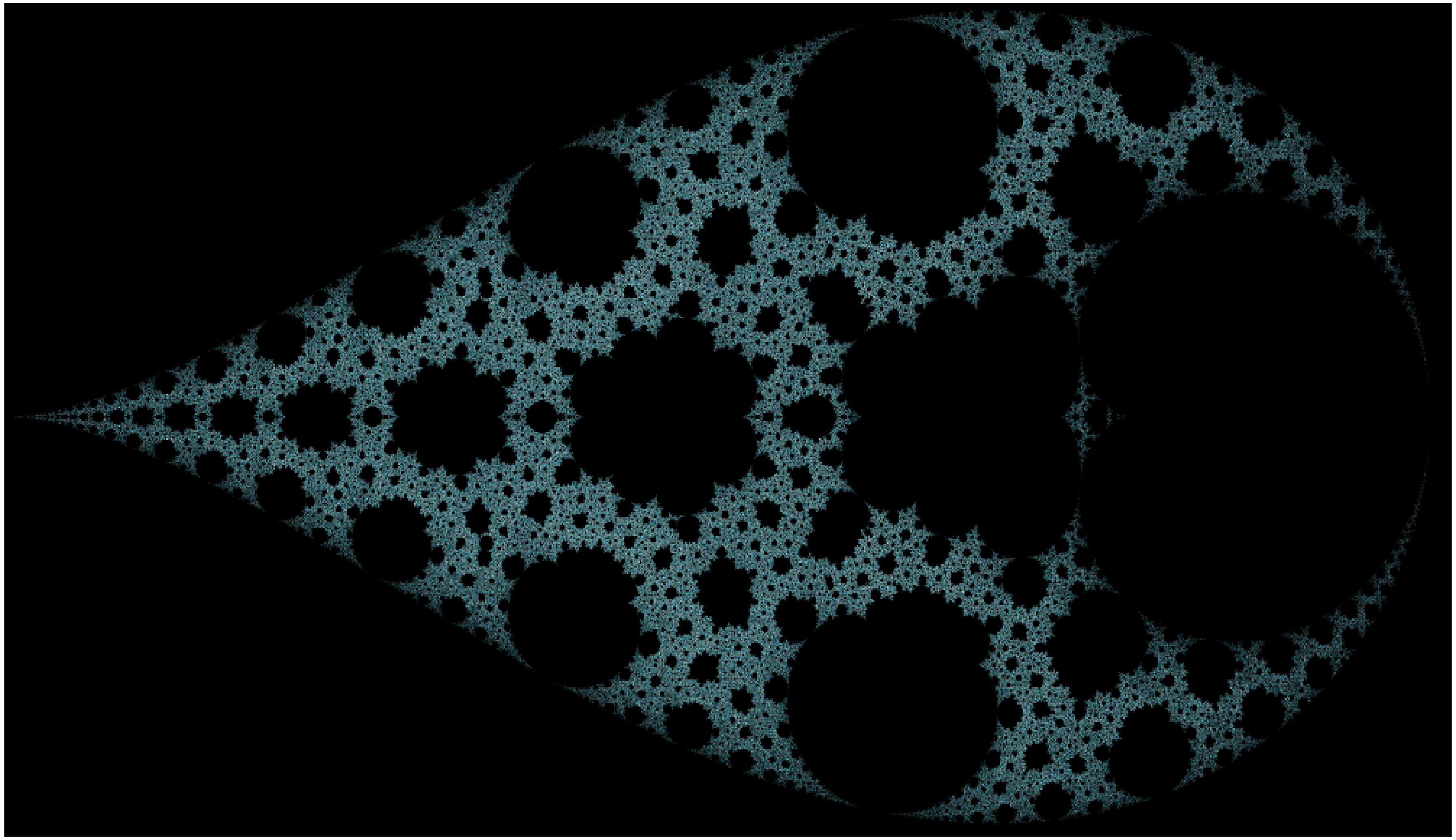
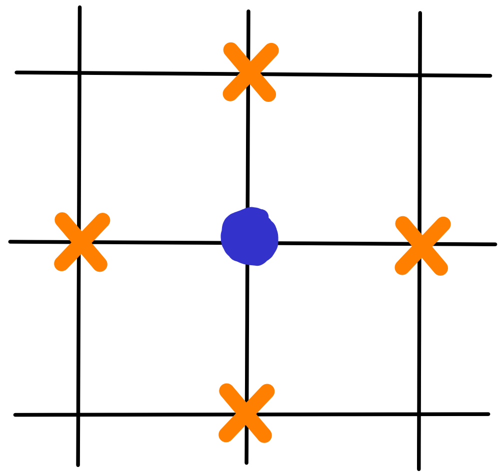


Figure by B. Espigule

The lattice gas hardcore model

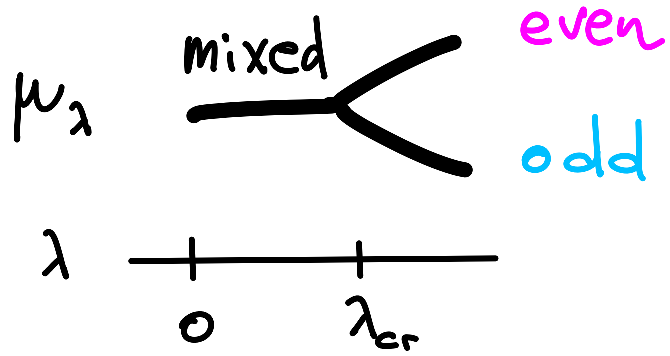


- Large particles on the vertices of $\mathcal{G}$ excluding other particles at the adjacent ones
- Activity parameter $\lambda \in [0, +\infty)$ determining the probability of particles appearing on vertices
- For $G \subset \mathcal{G}$ finite, the partition function is Z_G
- The pressure function is

$$\mathcal{P}(\lambda) = \lim_{G \to \mathcal{G}} \frac{1}{\# V_G} \log(Z_G(\lambda))$$

Phase transitions

Bifurcation of Gibbs measures



Related to non-analycity of the pressure function

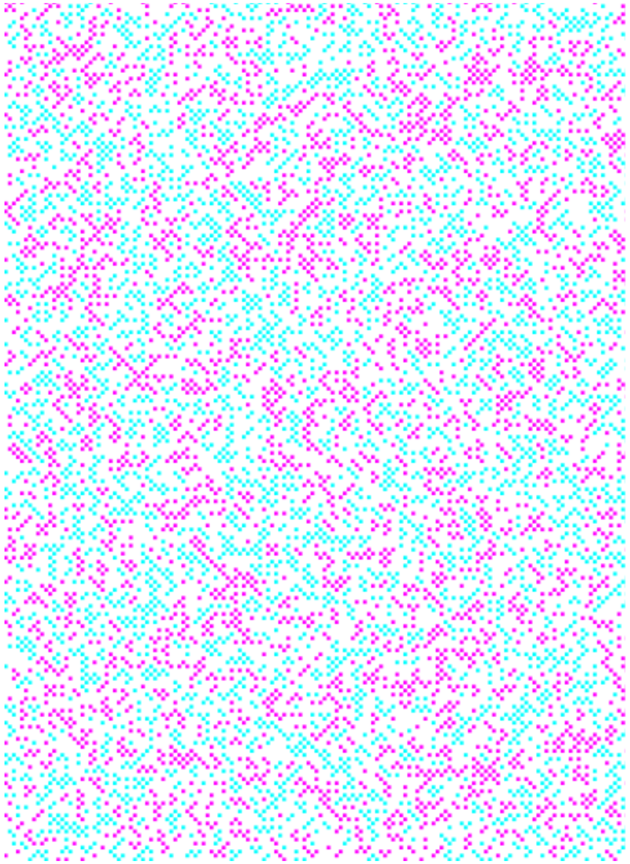
For the regular rooted tree Z:

$$\lambda_{cr} = \frac{d^d}{(d-1)^{d+1}}$$

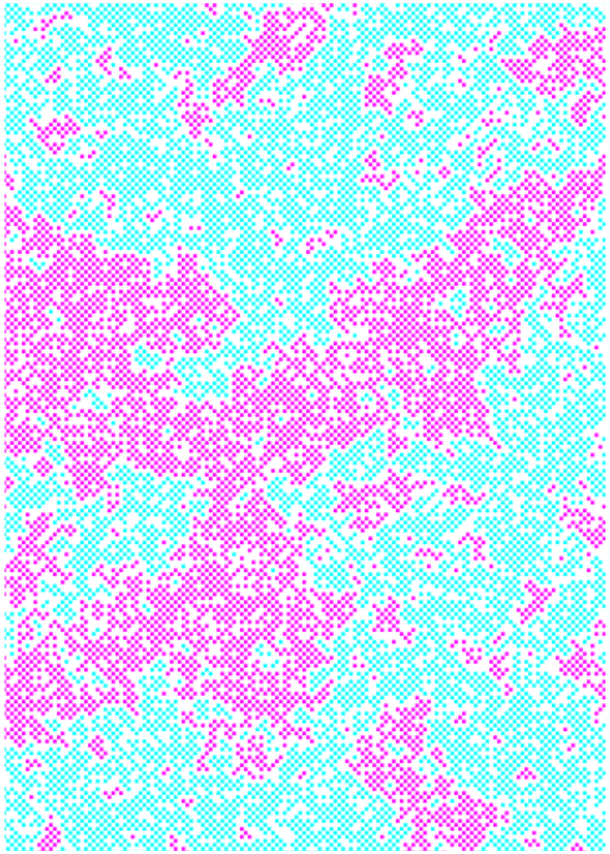
(Runnels 1967)

For $\mathbb{Z}^2$: $3.787 < \lambda_{cr} < 3.792$ (computationally!)

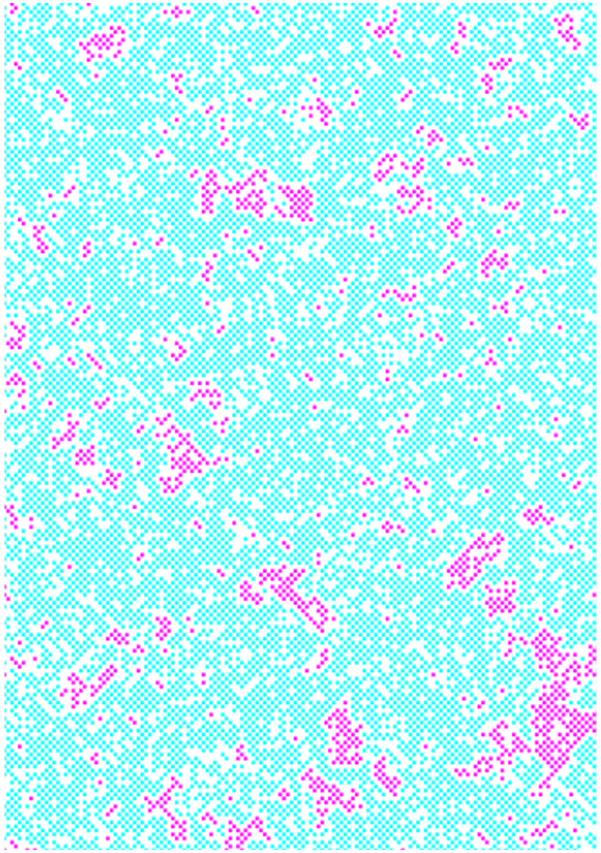
$\lambda = 1$



$\lambda = 3.787$



$\lambda = 3.792$



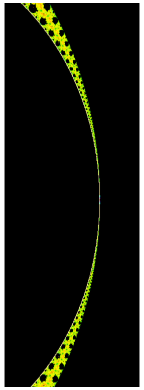
Figures by P. Shor & P. Winkler

THE PRESSURE FUNCTION IS SMOOTH

Set $\mathcal{S} = \mathcal{E}$ again. The QET implies that

$$\mathcal{P}(\lambda) = \int_{\mathbb{C}} \log|\lambda - \xi| \, d\mu_{bif}(\xi)$$

Used to prove:



PROP (RL-S) $\exists\, c > 0$ s.t.

$$\forall \varepsilon > 0 \quad \mu_{bif}\left(B(\lambda_{cr}, \varepsilon)\right) \leq e^{-c/\varepsilon}$$

THM (RL-S) $\mathcal{P}$ is analytic on $(0, +\infty) \setminus \{\lambda_{cr}\}$ and $\mathcal{C}^\infty$ at λ_{cr}

$\Rightarrow$ LGHM on $\mathcal{E}$ has a phase transition of "infinite order" at λ_{cr}

THANKS !